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On the representations of the group SU(3)

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Abstract. By a generalization of techniques developed earlier for dealing with the three-dimensional rotation group O_3 the generators of the group SU(3) are expressed as differential operators involving four independent variables. The reduction in the number of variables simplifies the mathematical problem and makes it easier to study the properties of the group and its irreducible representations. From the forms of the new operators it becomes apparent that the basic states of an irreducible representation of SU(3) are linear combinations of Clebsch–Gordan series of SU(2) (or O_3). A convenient expression for the latter, which simplifies the derivation and also brings out the significance of certain steps more clearly, is obtained here by a proper interpretation of the results given in an earlier paper by the author. Besides this certain recursion relations for the Clebsch–Gordan coefficients of SU(2) are also found to be helpful for studying the SU(3) representations. These relations are derived in a novel way from Gauss's relations between contiguous hypergeometric functions.

1. Introduction

The unitary unimodular group in three dimensions, the so-called SU(3) group, has assumed a position of paramount importance in the theory of strongly interacting elementary particles (Ikeda *et al.* 1959, Ne'eman 1961, Salam and Ward 1961, Gell-mann 1962, Matthews and Salam 1962) and in some other branches of physics (Elliott and Harvey 1963). The success in classifying the observed particles and resonances according to irreducible representations of this group has prompted physicists to study its mathematical properties in great detail (Behrends *et al.* 1962, Pursey 1963, Baird and Biedenharn 1963, de Swart 1963, Bég and Ruegg 1965, Sharp and von Baeyer 1966). As a consequence, a good understanding of the properties of the group and its irreducible representations has now been gained. In the present paper the main results on SU(3) are derived by a simpler method which requires very little of the machinery of the standard representation theory. The method is based on the author's work on the SU(2) group (or the rotation group in three dimensions) carried out in connection with a problem in molecular spectroscopy. In order to preserve continuity and make the treatment as self-contained as possible a brief survey of this work is given in the following paragraph.

In the first paper (Majumdar 1954) on the above-mentioned problem, some simplifications were introduced into the three-particle Hamiltonian by removing two of the Eulerian angles and expressing the angular momentum operators occurring in it in terms of a single variable ϕ, the angle of rotation about the moving z axis. This facilitated the calculation of the rotation–vibration energies of triatomic molecules, and enabled closed expressions to be derived for the various corrections to the energy. In spite of the advantages, however, the method had certain limitations and could not be extended, as such, to molecules containing more than three atoms. The extension needed a different approach and was made in a subsequent paper (Majumdar 1958 a). The Euler angles, which proved to be a hindrance rather than a help, were discarded and the angular momentum matrices were taken as the starting point for further elaboration of the technique. The matrices were first rationalized by a suitable similarity transformation and then represented as operators in a space spanned by the functions $\exp(im\phi)$ (m taking the usual values from $-j$ to $+j$). Besides providing a systematic basis for the previous work the new approach yielded an alternative set of operators which led to a convenient method (Majumdar 1958 b, to be referred to as I) of deriving the Clebsch–Gordan (CG) coefficients.† The reduction in

† CG coefficients of SU(2) alone will be considered in this paper.

the number of variables simplified the mathematical problem and made it possible to establish a connection between the coefficients and the Gauss hypergeometric function (HGF). As the transformation properties of the latter are well known this opened up the possibility of finding interesting relationships.

In view of the fruitfulness of the above approach it is pertinent to ask if a similar reduction could be carried out for the group SU(3). It is found that this is possible if we look at the question from a slightly different angle. The matrices for the generators of SU(3) are complicated and do not form a convenient starting point for the construction of operators of the above type. However, one can start from the differential operators which are much easier to handle. By using a basic property of the polynomials which transform according to an irreducible representation of the group these differential operators can be expressed, without imposing any restrictions on them, in terms of a lesser number of variables. To illustrate the procedure the case of SU(2) is considered first and the operators used in I are rederived from the new viewpoint. The same considerations applied to SU(3) give a set of operators which involve only four independent variables instead of the usual six. These operators are used in place of the customary ones in the discussions of the present paper. Expressed in terms of them, the eigenvalue equation of the quadratic Casimir operator takes the form of a partial differential equation in four independent variables. The polynomial solutions of this equation form the basis of an irreducible representation of SU(3).

It is known, and can also be inferred from the forms of the operators (10*a*)–(10*d*) of § 3, that a basic state of an irreducible representation of SU(3) is a linear combination of several CG series with the same values of j, m and j_1-j_2. The problem of determining the basic states is thus closely connected with the theory of coupling of two angular momenta. As already indicated, this theory was treated in I by the new technique and the coupled states were obtained as functions of two hypothetical variables ϕ_1 and ϕ_2. By assigning a meaning to these variables it has now been possible to go a step further and write the CG series itself in a compact form suitable for application to the problem under investigation. The new expression for the CG series (which does not appear to have been recorded in the literature) is easily derived from the function χ_m of I (see the appendix) and contains a HGF, with parameters $a = -j-m$, $b = j_1-j_2-j$ and $c = -2j$, as a factor. For fixed values of j, m and j_1-j_2, this HGF, together with certain other quantities, separates out as a common factor from the series (14) of § 4 defining a basic state, while the coefficients

$$A_{j_1j_2jm\delta}C_{j_1j_2j}(-x\bar{x}-y\bar{y})^{j_1+j_2-j}$$

with the appropriate values of $A_{j_1j_2jm\delta}$ sum up to another HGF. A basic state is thus obtained, apart from a simple functional factor, as a product of two HGF's. This result has been obtained already, though not in the same form, by a different method (Bég and Ruegg 1965) which essentially consists in constructing the eigenfunctions of the Laplace–Beltrami operator on a hypersphere.

Finally, we briefly mention the other results given in the paper. Section 5 contains a simple derivation of the matrix elements of the generators in an arbitrary representation. In the appendix are derived certain relations between contiguous CG coefficients required in §§ 4 and 5. The relations are, of course, not all new, but the use of the HGF for obtaining them is an interesting feature of the present treatment.

2. Analytical operators for the generators of SU(2)

An element of the unitary unimodular group SU(2) may be regarded as the matrix of the linear transformation of a pair of complex variables ξ and η. The generators j_x, j_y, j_z of this group are multiples of the Pauli spin matrices, and are equivalent to the differential operators

$$j_x = \tfrac{1}{2}(\eta\partial_\xi+\xi\partial_\eta), \qquad j_y = \tfrac{1}{2}i(\eta\partial_\xi-\xi\partial_\eta), \qquad j_z = \tfrac{1}{2}(\xi\partial_\xi-\eta\partial_\eta). \tag{1}$$

As is well known, a $(2j+1)$-dimensional representation† of SU(2) is obtained from the

† The word 'representation' will, generally, mean a 'unitary irreducible representation'.

transformation of a symmetric tensor of rank $2j$ constructed from the vector (ξ, η). Since the components of this tensor span a space of homogeneous polynomials of degree $2j$, it is possible to write the operators in the simpler forms

$$j_+ = j_x + ij_y = 2jx - x^2\partial_x, \qquad j_- = j_x - ij_y = \partial_x, \qquad j_z = x\partial_x - j \tag{2}$$

where $x = \xi/\eta$. These operators occur in Naimark's treatment (Naimark 1964) of the three-dimensional rotation group. They are, however, equivalent to the operators constructed by the present author from very different considerations. To see this we pass on to a new basis by means of the transformation $x^{-j} j_{\pm,z} x^j$. This gives

$$j_\pm = x^{\pm 1}(j \mp x\partial_x), \qquad j_z = x\partial_x.$$

These operators go over into the forms given in equation (1) of I (or equation (22) of Majumdar 1958 a) when x is replaced by $\exp(i\phi)$. The equivalence of the two sets of operators is thus established. The analysis also clarifies the meaning of the variable ϕ by relating it to the vector (ξ, η) of the linear space on which the matrices of SU(2) operate.

We can now put the eigenfunctions of the product representation into a more convenient form. These eigenfunctions were obtained in I in several equivalent forms†, one of which is

$$\Phi_m = \exp(im\phi_1)\chi_m = \exp(im\phi_1) C_{j_1 j_2 j} \chi^{-j_2}(1-\chi)^{j_1+j_2-j} F(-j-m, j_1-j_2-j, -2j, 1-\chi) \tag{3}$$

where $\chi = \exp i(\phi_2 - \phi_1)$ and

$$C_{j_1 j_2 j} = \left\{\frac{(2j)!\,(2j+1)!}{(j_1+j_2-j)!\,(j_1-j_2+j)!\,(-j_1+j_2+j)!\,(j_1+j_2+j+1)!}\right\}^{1/2}. \tag{4}$$

With the above interpretation of ϕ this can be written as

$$\Phi_m = \left(\frac{\xi_1}{\eta_1}\right)^m C_{j_1 j_2 j}\left(\frac{\xi_2\eta_1}{\eta_2\xi_1}\right)^{-j_2}\left(\frac{\eta_2\xi_1 - \xi_2\eta_1}{\eta_2\xi_1}\right)^{j_1+j_2-j} F\left(-j-m, j_1-j_2-j, -2j, 1-\frac{\xi_2\eta_1}{\eta_2\xi_1}\right) \tag{5}$$

which is a function of degree zero in the variables ξ_1, η_1, ξ_2 and η_2. Multiplying it by

$$\{(j+m)!\,(j-m)!\}^{-1/2}(\xi_1\eta_1)^{j_1}(\xi_2\eta_2)^{j_2}$$

we obtain the CG series

$$\left|\begin{matrix} j_1 & j_2 & j \\ & & m \end{matrix}\right\rangle = \{(j+m)!\,(j-m)!\}^{-1/2} C_{j_1 j_2 j}\, \xi_1^{j_1-j_2+m} \eta_1^{j_1-j_2-m} (\eta_2\xi_1)^{-j_1+j_2+j}$$
$$\times (\eta_2\xi_1 - \xi_2\eta_1)^{j_1+j_2-j} F\left(-j-m, j_1-j_2-j, -2j, 1-\frac{\xi_2\eta_1}{\eta_2\xi_1}\right). \tag{6}$$

With the aid of the relation between the CG coefficients and the Taylor coefficients of the function χ_m of equation (3) this can be brought into the form

$$\left|\begin{matrix} j_1 & j_2 & j \\ & & m \end{matrix}\right\rangle = \sum_{m_2} \begin{Bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{Bmatrix} \{(j_1+m_1)!\,(j_1-m_1)!\,(j_2+m_2)!\,(j_2-m_2)!\}^{-1/2}$$
$$\times \xi_1^{j_1+m_1} \eta_1^{j_1-m_1} \xi_2^{j_2+m_2} \eta_2^{j_2-m_2}. \tag{7}$$

† The various forms are connected by Kummer's relations, which must be used with caution when the parameters of the HGF are integers. The four different forms of χ_m were obtained in I without using Kummer's relations and are *always valid for physical values of* j_1, j_2, j, m. But for non-physical values of these quantities the situation may be otherwise. Let us consider, for instance, the relation

$$F(-j-m, \delta-j; -2j; z) = (1-z)^{-\delta+m} F(-j+m, -\delta-j; -2j; z).$$

For $\delta < -j$, and j, m physical (or for $m > j$, and j, δ physical) the highest power of z on the left-hand side of the above equation is not greater than $2j$, while it is greater than $2j$ on the right-hand side, and this is a contradiction.

Both the expressions (6) and (7) will be needed for the subsequent discussions. The expression (6) does not contain the CG coefficient explicitly, and proves indispensable for the development of § 4. In using the expression (7) we shall find it convenient to replace the CG coefficient $\begin{Bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{Bmatrix}$ occurring in it by the 'coefficient of the second kind' (cf. § 2 of I), defined by the equation

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} = \begin{Bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{Bmatrix} \{(j+m)!\,(j-m)!\}^{1/2} \times \{(j_1+m_1)!\,(j_1-m_1)!\,(j_2+m_2)!\,(j_2-m_2)!\}^{-1/2}. \tag{8}$$

The results can be expressed more economically in terms of the latter.

3. Analytical operators for the generators of SU(3)

In extending the above considerations to the group SU(3) we have to consider both the defining representation (1, 0) and its conjugate (0, 1). The matrices of these fundamental representations operate on the linear spaces of the complex vectors (ξ, η, ζ) and $(\bar\xi, \bar\eta, \bar\zeta)$ which are supposed to be completely independent of each other. In terms of these six variables the eight generators of the group can be written as differential operators. The matrix forms of these generators, with six of them combined into raising and lowering operators, are given in the important article by Behrends *et al.* (1962). The corresponding differential operators are

$$\begin{aligned} I_- &= \sqrt{6}E_{-1} = \eta\partial_\xi - \bar\xi\partial_{\bar\eta}, \qquad I_+ = \sqrt{6}E_1 = \xi\partial_\eta - \bar\eta\partial_{\bar\xi} \\ 2I_z &= 2\sqrt{3}H_1 = \xi\partial_\xi - \eta\partial_\eta - \bar\xi\partial_{\bar\xi} + \bar\eta\partial_{\bar\eta} \\ 6H_2 &= \xi\partial_\xi + \eta\partial_\eta - 2\zeta\partial_\zeta - \bar\xi\partial_{\bar\xi} - \bar\eta\partial_{\bar\eta} + 2\bar\zeta\partial_{\bar\zeta} \\ \sqrt{6}E_{-2} &= \zeta\partial_\xi - \bar\xi\partial_{\bar\zeta}, \qquad \sqrt{6}E_2 = \xi\partial_\zeta - \bar\zeta\partial_{\bar\xi} \\ \sqrt{6}E_{-3} &= \zeta\partial_\eta - \bar\eta\partial_{\bar\zeta}, \qquad \sqrt{6}E_3 = \eta\partial_\zeta - \bar\zeta\partial_{\bar\eta}. \end{aligned} \tag{9}$$

These operators have been used in the past (Bég and Ruegg 1965, Sharp and von Baeyer 1966) for studying the properties of the group. In the present investigation, however, we find it convenient to use a different set of operators which can be obtained by the following considerations.

A basic state of a representation (p, q) is a homogeneous polynomial of degree p in ξ, η, ζ and of degree q in $\bar\xi, \bar\eta, \bar\zeta$. After division by $\zeta^p\bar\zeta^q$ this can be written as an inhomogeneous polynomial of degree not exceeding p in $x = \xi/\zeta$, $y = \eta/\zeta$, and of degree not exceeding q in $\bar x = \bar\xi/\bar\zeta$, $\bar y = \bar\eta/\bar\zeta$. As in the case of SU(2), the operators then take the simpler forms

$$I_- = \sqrt{6}E_{-1} = y\partial_x - \bar x\partial_{\bar y} \tag{10a}$$

$$I_+ = \sqrt{6}E_1 = x\partial_y - \bar y\partial_{\bar x} \tag{10b}$$

$$2I_z = 2\sqrt{3}H_1 = x\partial_x - y\partial_y - \bar x\partial_{\bar x} + \bar y\partial_{\bar y} \tag{10c}$$

$$Y = 2H_2 = \tfrac{1}{3}(p-q) - P + Q \tag{10d}$$

$$\sqrt{6}E_{-2} = \partial_x - \bar x Q \tag{10e}$$

$$\sqrt{6}E_2 = -\partial_{\bar x} + xP \tag{10f}$$

$$\sqrt{6}E_{-3} = \partial_y - \bar y Q \tag{10g}$$

$$\sqrt{6}E_3 = -\partial_{\bar y} + yP \tag{10h}$$

where $P = p - x\partial_x - y\partial_y$, $Q = q - \bar x\partial_{\bar x} - \bar y\partial_{\bar y}$.

These operators involve only four independent variables instead of six and are much more convenient to work with than the previous ones. To verify that they obey the usual commutation rules we make use of certain symmetry properties, which are easily found out by inspection. Denoting an interchange by an arrow pointing both ways, we observe that

(i) $(x \leftrightarrow -\bar{x},\ y \leftrightarrow -\bar{y},\ p \leftrightarrow q)$ lead to $E_1 \leftrightarrow -E_{-1},\ E_2 \leftrightarrow E_{-2},\ E_3 \leftrightarrow E_{-3}$

(ii) $(x \leftrightarrow -\bar{y},\ y \leftrightarrow -\bar{x},\ p \leftrightarrow q)$ lead to $E_{\pm 1} \leftrightarrow -E_{\pm 1},\ E_2 \leftrightarrow E_{-3},\ E_{-2} \leftrightarrow E_3$

(iii) $(x \leftrightarrow y,\ \bar{x} \leftrightarrow \bar{y})$ lead to $E_1 \leftrightarrow E_{-1},\ E_2 \leftrightarrow E_3,\ E_{-2} \leftrightarrow E_{-3}$.

Under all these interchanges the operators P and Q either remain unchanged or transform into one another. These symmetry properties, of which only two are independent, considerably lighten the algebra at certain stages of the calculation.

As an illustration let us calculate the quadratic Casimir operator

$$C = H_1^2 + H_2^2 + \sum_{\alpha > 0} (E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha)$$

in terms of the new variables. Starting from $E_1 E_{-1}$, $E_2 E_{-2}$, and using the symmetry properties we obtain the expression for

$$\sum_{\alpha > 0} (E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha).$$

When the contributions due to H_1 and H_2 are taken into account, the operator takes the form

$$C = \tfrac{1}{9}(p^2 + q^2 + pq + 3p + 3q) - \tfrac{1}{3}(x\bar{x} + y\bar{y} + 1)(\partial_{x\bar{x}} + \partial_{y\bar{y}} + PQ). \tag{11}$$

The eigenfunctions of C with the eigenvalue $\frac{1}{9}(p^2 + q^2 + pq + 3p + 3q)$ are the basic states occurring in the representation (p, q). Since the eigenvalue is equal to the first term in the expression for C, the eigenfunctions can be determined by solving the equation

$$(\partial_{x\bar{x}} + \partial_{y\bar{y}} + PQ) f = 0. \tag{12}$$

This is a partial differential equation of a most inconvenient type, but the task of solving it is greatly simplified by taking f to be a simultaneous eigenfunction of H_1, H_2 and $H_1^2 + E_1 E_{-1} + E_{-1} E_1$. This choice leads to a representation which is explicitly reduced with respect to the subgroup of isotopic spin.

4. Determination of the basic states

From the remarks made at the end of the last section it is clear that a basic state belonging to a representation (p, q) is a linear combination of several CG series with the same values of j, m, $j_1 - j_2$. The numbers j and m are identified with the isotopic spin and its z components in physical applications, and $\delta \equiv j_1 - j_2$ is related to the hypercharge Y by the equation

$$Y = 2(j_1 - j_2) - \tfrac{2}{3}(p - q). \tag{13}$$

Using the two alternative forms of the CG series given by equations (6) and (7), and denoting a basic state by the symbol $|jm\delta\rangle$, we can therefore write

$$f \equiv |jm\delta\rangle = \sum_{j_2} A_{j_1 j_2 j m \delta} C_{j_1 j_2 j} x^{\delta + m} y^{\delta - m} (-x\bar{x})^{-\delta + j} (-x\bar{x} - y\bar{y})^{j_1 + j_2 - j}$$

$$\times F\left(-j - m, \delta - j, -2j, 1 + \frac{y\bar{y}}{x\bar{x}}\right) \tag{14}$$

$$= \sum_{j_2 m_2} A_{j_1 j_2 j m \delta} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} x^{j_1 + m_1} y^{j_1 - m_1} \bar{y}^{j_2 + m_2} (-\bar{x})^{j_2 - m_2}$$

$$\equiv \sum_{j_2 m_2} A_{j_1 j_2 j m \delta} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix} |j_1 m_1, j_2 m_2\rangle. \tag{15}$$

The coefficients $A_{j_1 j_2 j m \delta}$ occurring here are to be determined from the condition that $|jm\delta\rangle$ is a solution of equation (12). Since j, m, δ have fixed values for the various CG series giving rise to a particular state, we can take out a common factor and write the expression (14) in the form

$$f = x^{\delta+m}y^{\delta-m}(-x\bar{x})^{-\delta+j}F\left(-j-m, \delta-j, -2j, 1+\frac{y\bar{y}}{x\bar{x}}\right)g(x\bar{x}+y\bar{y}).$$

This suggests a transformation to the variables

$$\lambda = x, \qquad \rho = y, \qquad u = x\bar{x}+y\bar{y}, \qquad v = 1+\frac{y\bar{y}}{x\bar{x}}. \tag{16}$$

In terms of the new variables equation (12) becomes

$$\{\lambda\partial_{u\lambda}+\rho\partial_{u\rho}+(1-v)u^{-1}v\lambda\partial_{v\lambda}+u^{-1}\rho v\partial_{v\rho}+u\partial_{uu}+2\partial_u$$
$$+u^{-1}(v-1)v^2\partial_{vv}+u^{-1}v^2\partial_v+(p-\lambda\partial_\lambda-\rho\partial_\rho-u\partial_u)(q-u\partial_u)\}f = 0. \tag{17}$$

The solution is

$$f = \text{const.}\times\lambda^{\delta+m}\rho^{\delta-m}\left(-\frac{u}{v}\right)^{-\delta+j}F(\delta+j-p, -\delta+j-q, 2j+2, -u)$$
$$\times F(-j-m, \delta-j, -2j, v). \tag{18}$$

The above solution can also be obtained by substituting the expression (15) in equation (12) and using relation (A9) of the appendix. This leads to the recurrence relation

$$A_{j_1+\frac{1}{2},j_2+\frac{1}{2},jm\delta}(J-2j+1)^{1/2}(J+2)^{1/2} = (p-2j_1)(q-2j_2)A_{j_1j_2jm\delta} \tag{19}$$

for two successive coefficients of the linear combination occurring in (15). From this the general form of the coefficient is easily seen to be

$$A_{j_1j_2jm\delta} = A_{jm\delta}\{(J-2j)!\,(J+1)!\}^{-1/2}\{(p-2j_1)!\,(q-2j_2)!\}^{-1}. \tag{20}$$

Inserting this in equation (15), we have

$$|jm\delta\rangle = A_{jm\delta}\|jm\delta\rangle\!\rangle = A_{jm\delta}\{(J-2j)!\,(J+1)!\}^{-1/2}\{(p-2j_1)!\,(q-2j_2)!\}^{-1}$$
$$\times\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m\end{pmatrix}|j_1m_1, j_2m_2\rangle \tag{21}$$

where the values of j_1 and j_2 lie within the limits $j \leqslant j_1+j_2$, $p \geqslant 2j_1$, $q \geqslant 2j_2$. This alternative form of the solution is equally convenient for practical applications and will be used in § 5 for deriving the matrices of the generators in an arbitrary representation. It is easily transformed into the previous form (18) by using the expression (6) for the CG series and carrying out the summation over j_2. With the value of $A_{j_1j_2jm\delta}$ just found the sum reduces to another HGF, and we have

$$|jm\delta\rangle = A_{jm\delta}\{(2j)!\}^{1/2}\{(2j+1)!\,(\delta+j)!\,(-\delta+j)!\}^{-1/2}\{(-\delta-j+p)!\,(\delta-j+q)!\}^{-1}$$
$$\times x^{\delta+m}y^{\delta-m}(-x\bar{x})^{-\delta+j}F(\delta+j-p, -\delta+j-q, 2j+2, -x\bar{x}-y\bar{y})$$
$$\times F\left(-j-m, \delta-j, -2j, 1+\frac{y\bar{y}}{x\bar{x}}\right). \tag{22}$$

5. The matrices of the generators

Under transformation of the isotopic spin subgroup generated by H_1, $E_{\pm 1}$ the operators E_2, E_3 transform like the $\pm\frac{1}{2}$ components of an irreducible tensor (bispinor) operator $T_{1/2}{}^r$ of rank $\frac{1}{2}$, while $-E_{-3}$, E_{-2} transform like the $\pm\frac{1}{2}$ components of the Hermitian adjoint operator $\bar{T}_{1/2}{}^r$. Using the Wigner–Eckart theorem, it is therefore possible to write the matrix elements in the factorized form

$$\langle j'm'\delta'|T_{1/2}{}^r|jm\delta\rangle = (-)^{j'-m'}\begin{bmatrix} j' & \frac{1}{2} & j \\ -m' & r & m\end{bmatrix}\langle j'\delta'\|T\|j\delta\rangle$$

where $\langle j'\delta'\|T\|j\delta\rangle$ is the reduced matrix element. Starting from these considerations

Pursey (1963) and Baird and Biedenharn (1963) independently derived the matrix forms of $E_{\pm 2}$, $E_{\pm 3}$ in an arbitrary representation. We shall see, however, that the same results can be obtained more easily by using the relations between contiguous CG coefficients.

By equation (A6*a*) of the appendix

$$xP\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix}|j_1m_1, j_2m_2\rangle = \left[\left\{\frac{(J-2j_2+1)(J+2)}{(2j+1)(2j+2)}\right\}^{1/2}(A) + (j-m)\left\{\frac{(J-2j+1)(J-2j_1)}{2j(2j+1)}\right\}^{1/2}(B)\right](p-2j_1)|j_1+\tfrac{1}{2}\, m_1+\tfrac{1}{2}, j_2m_2\rangle \tag{23}$$

and by equation (A8*a*)

$$-\partial_{\bar{x}}\begin{pmatrix} j_1+\frac{1}{2} & j_2+\frac{1}{2} & j \\ m_1+\frac{1}{2} & m_2-\frac{1}{2} & m \end{pmatrix}|j_1+\tfrac{1}{2}\, m_1+\tfrac{1}{2}, j_2+\tfrac{1}{2}\, m_2-\tfrac{1}{2}\rangle = \left[\left\{\frac{(J-2j_2+1)(J-2j+1)}{(2j+1)(2j+2)}\right\}^{1/2}(A) + (j-m)\left\{\frac{(J-2j_1)(J+2)}{2j(2j+1)}\right\}^{1/2}(B)\right]|j_1+\tfrac{1}{2}\;\; m_1+\tfrac{1}{2}, j_2m_2\rangle. \tag{24}$$

From equations (10*f*), (21), (23) and (24) we have

$$\sqrt{6}E_2\|jm\delta\rangle\!\rangle = \left\{\frac{\delta+j+1}{(2j+1)(2j+2)}\right\}^{1/2}(\delta+j+2+q)\|j+\tfrac{1}{2}, m+\tfrac{1}{2}, \delta+\tfrac{1}{2}\rangle\!\rangle + (j-m)\left\{\frac{-\delta+j}{2j(2j+1)}\right\}^{1/2}(\delta-j+1+q)\|j-\tfrac{1}{2}, m+\tfrac{1}{2}, \delta+\tfrac{1}{2}\rangle\!\rangle. \tag{25}$$

Analogous formulae for $\sqrt{6}(E_{-2}, E_3, E_{-3})\|jm\delta\rangle\!\rangle$ are easily obtained from this by using the symmetry properties discussed in § 3. In case (i), for instance, the simultaneous interchanges $x \leftrightarrow -\bar{x}$, $y \leftrightarrow -\bar{y}$, $p \leftrightarrow q$ transform E_2 into E_{-2}, and $\|jm\delta\rangle\!\rangle$ into $(-)^{-\delta+m}\|j\; -m\; -\delta\rangle\!\rangle$. After this symmetry operation the matrices of E_2 and E_{-2} are easily determined from the condition that they must be Hermitian conjugates of each other in a unitary representation of the group. In cases (ii) and (iii) the function $\|jm\delta\rangle\!\rangle$ transforms into $(-)^{j+m}\|j\; m\; -\delta\rangle\!\rangle$ and $(-)^{-\delta+j}\|j\; -m\; \delta\rangle\!\rangle$ respectively. These symmetry operations enable us to determine the matrices E_3 and E_{-3}.

Appendix. Relations between contiguous CG coefficients

It was shown in I that the CG coefficient of the second kind defined by equation (8) is the coefficient of x^{m_2} in the Taylor expansion of the function

$$\chi_m = C_{j_1j_2j}x^{-j_2}(1-x)^{j_1+j_2-j}F(-j-m, j_1-j_2-j, -2j, 1-x). \tag{A1}$$

The occurrence of the HGF in the expression for χ_m has interesting consequences. The well-known transformation properties of this function enable χ_m to be written in a number of different forms, each form leading to a different expression for the CG coefficient. Secondly, for every relation between contiguous HGF's (Magnus and Oberhettinger 1948, Erdélyi 1953) there exists a relation between contiguous CG coefficients with j values differing by 0, $\pm\frac{1}{2}$. However, like the various expressions for the CG coefficients these relations are also not all independent and can be obtained, after symmetry operations, as linear combinations of only a few of them. Some of these relations, particularly, those that were required in the preceding sections, will be derived here by using the properties of the HGF. A list of the formulae and symbols needed for this purpose is given below.

For the CG coefficients occurring most frequently we shall use the abbreviations

$$(K) = \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix}, \qquad (C) = \begin{pmatrix} j_1+\frac{1}{2} & j_2+\frac{1}{2} & j \\ m_1+\frac{1}{2} & m_2-\frac{1}{2} & m \end{pmatrix}$$

$$(A) = \begin{pmatrix} j_1+\frac{1}{2} & j_2 & j+\frac{1}{2} \\ m_1+\frac{1}{2} & m_2 & m+\frac{1}{2} \end{pmatrix}, \qquad (B) = \begin{pmatrix} j_1+\frac{1}{2} & j_2 & j-\frac{1}{2} \\ m_1+\frac{1}{2} & m_2 & m+\frac{1}{2} \end{pmatrix}.$$

The corresponding 3-j symbols will be denoted by

$$[K] = \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}, \text{ etc.,}$$

where $j_3 = j$, $m_3 = -m$, and $m_1+m_2+m_3 = 0$. The greater symmetry of the 3-j symbol facilitates passage from one relation to another.

The HGF $F(a, b, c, z)$ will be denoted by F and the contiguous functions $F(a-1, b, c, z)$, $F(a, b+1, c, z)$, ... by $F(a-1)$, $F(b+1)$, ...

We shall also require the well-known recurrence formulae connecting CG coefficients with the same j but different m values (cf. equations (27) and (28) of Majumdar 1958 a). These are

$$\begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2+1 & m+1 \end{pmatrix}(j-m) = \begin{pmatrix} j_1 & j_2 & j \\ m_1-1 & m_2+1 & m \end{pmatrix}(j_1-m_1+1)+(K)(j_2-m_2) \quad \text{(A2)}$$

$$\begin{aligned} 0 = & \begin{pmatrix} j_1 & j_2 & j \\ m_1+1 & m_2-1 & m \end{pmatrix}(j_1+m_1+1)(j_2-m_2+1) \\ & +(K)\{j_1(j_1+1)+j_2(j_2+1)-j(j+1)+2m_1m_2\} \\ & +\begin{pmatrix} j_1 & j_2 & j \\ m_1-1 & m_2+1 & m \end{pmatrix}(j_1-m_1+1)(j_2+m_2+1). \end{aligned} \quad \text{(A3)}$$

Let us consider now the function

$$\chi_m(1-x). \quad \text{(A4)}$$

According to the theorem stated in the beginning of this section, the coefficient of x^{m_2} in the Taylor expansion of this function is

$$(K)-\begin{pmatrix} j_1 & j_2 & j \\ m_1+1 & m_2-1 & m \end{pmatrix}.$$

Changing j_1 to $j_1+\frac{1}{2}$, j_2 to $j_2+\frac{1}{2}$ in the function (A4) and keeping j and m unchanged we see that this is also equal to

$$\frac{(C)C_{j_1j_2j}}{C_{j_1+\frac{1}{2},j_2+\frac{1}{2},j}}.$$

Therefore, by equation (4),

$$(K)-\begin{pmatrix} j_1 & j_2 & j \\ m_1+1 & m_2-1 & m \end{pmatrix} = (C)(J-2j+1)^{1/2}(J+2)^{1/2} \quad \text{(A5}a\text{)}$$

where $J = j_1+j_2+j_3$. In terms of the 3-j symbols this can be written as

$$\begin{aligned} [K](j_1+m_1+1)^{1/2}(j_2-m_2+1)^{1/2} - \begin{bmatrix} j_1 & j_2 & j_3 \\ m_1+1 & m_2-1 & m_3 \end{bmatrix}(j_1-m_1)^{1/2}(j_2+m_2)^{1/2} \\ = [C](J-2j_3+1)^{1/2}(J+2)^{1/2}. \end{aligned} \quad \text{(A5}b\text{)}$$

After simple manipulation this gives a relation which is deduced by Edmonds from the properties of the 6-j symbols (cf. Edmonds 1957, equation (3.7.12), chap. 3).

Next, let us consider the relation

$$F = F(a-1, c-1)+\frac{b(c-a)}{c(c-1)}zF(b+1, c+1)$$

with $a = -j-m$, $b = j_1-j_2-j$, $c = -2j$, $z = 1-x$. Multiplying it by $C_{j_1j_2j}z^{J-2j}$ and equating the coefficients of $x^{j_2+m_2}$ we have

$$(K) = (A)\left\{\frac{(J-2j_2+1)(J+2)}{(2j+1)(2j+2)}\right\}^{1/2}+(B)(j-m)\left\{\frac{(J-2j+1)(J-2j_1)}{2j(2j+1)}\right\}^{1/2} \quad \text{(A6}a\text{)}$$

or, in terms of the 3-j symbols,

$$0 = [K](2j_3+1)(j_1+m_1+1)^{1/2}+[A]\{(j_3-m_3+1)(J-2j_2+1)(J+2)\}^{1/2} + [B]\{(j_3+m_3)(J-2j_3+1)(J-2j_1)\}^{1/2}. \tag{A6b}$$

Changing the signs of m_1, m_2, m_3 in this equation, using the symmetry of the 3-j symbol for reversal of signs, changing m_1 to m_1+1, m_3 to m_3-1, and using (A2), we have

$$(2j_3+1)\{(j_1-m_1)(j_2+m_2)(j_2-m_2+1)\}^{1/2}\begin{bmatrix} j_1 & j_2 & j_3 \\ m_1+1 & m_2-1 & m_3\end{bmatrix} = -[A](j_3-j_1-m_2)\{(j_3-m_3+1)(J-2j_2+1)(J+2)\}^{1/2} + [B](j_3+j_1+m_2+1)\{(j_3+m_3)(J-2j_3+1)(J-2j_1)\}^{1/2}. \tag{A7}$$

From (A5b), (A6b) and (A7) follows the relation

$$0 = (2j_3+1)(j_2-m_2+1)^{1/2}[C]+[A]\{(j_3-m_3+1)(J-2j_2+1)(J-2j_3+1)\}^{1/2} + [B]\{(j_3+m_3)(J-2j_1)(J+2)\}^{1/2} \tag{A8b}$$

or the equivalent relation

$$(j_2-m_2+1)(C) = (A)\left\{\frac{(J-2j_2+1)(J-2j+1)}{(2j+1)(2j+2)}\right\}^{1/2} + (B)(j-m)\left\{\frac{(J-2j_1)(J+2)}{2j(2j+1)}\right\}^{1/2}. \tag{A8a}$$

Another useful relation can be obtained by eliminating $\begin{pmatrix} j_1 & j_2 & j \\ m_1\mp1 & m_2\pm1 & m\end{pmatrix}$ from (A3), (A5a), and from the relation obtained from (A5a) by changing m_1 to m_1-1 and m_2 to m_2+1. The result is

$$-(C)(j_1+m_1+1)(j_2-m_2+1)+\begin{pmatrix} j_1+\frac{1}{2} & j_2+\frac{1}{2} & j \\ m_1-\frac{1}{2} & m_2+\frac{1}{2} & m\end{pmatrix}(j_1-m_1+1)(j_2+m_2+1) = -(J-2j+1)^{1/2}(J+2)^{1/2}(K). \tag{A9}$$

The above relations are sufficient for deriving the results of the preceding sections. However, a few more are given below as further illustrations of the use of the HGF in the present problem.

(i) $(b-c+1)F+(c-1)F(a-1,\ c-1)-b(1-z)F(b+1) = 0$ gives

$$0 = (K)(J-2j_1+1)^{1/2}(2j+2)^{1/2}-\begin{pmatrix} j_1 & j_2+\frac{1}{2} & j+\frac{1}{2} \\ m_1 & m_2-\frac{1}{2} & m-\frac{1}{2}\end{pmatrix}(J+2)^{1/2}(2j+1)^{1/2} + \begin{pmatrix} j_1-\frac{1}{2} & j_2+\frac{1}{2} & j \\ m_1+\frac{1}{2} & m_2-\frac{1}{2} & m\end{pmatrix}(J-2j_2)^{1/2}(2j+2)^{1/2}. \tag{A10}$$

(ii) $F-F(a+1)+(b/c)zF(a+1,\ b+1,\ c+1) = 0$, combined with (A2), gives

$$0 = (K)(j-j_2-m_1)(2j)^{1/2}-\begin{pmatrix} j_1 & j_2 & j \\ m_1-1 & m_2+1 & m\end{pmatrix}(j_1-m_1+1)(2j)^{1/2} + \begin{pmatrix} j_1 & j_2+\frac{1}{2} & j-\frac{1}{2} \\ m_1 & m_2+\frac{1}{2} & m+\frac{1}{2}\end{pmatrix}(j-m)\{(2j+1)(J-2j+1)(J-2j_2)\}^{1/2}. \tag{A11}$$

(iii) $z\,dF/dz+aF = aF(a+1)$, multiplied by $C_{j_1j_2j}z^{J-2j}$, gives (after symmetry operations) the recurrence formula (A2).

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